

Distance Regular Colorings of n -Dimensional Rectangular Grid

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Abstract. We study the infinite graph of n -dimensional rectangular grid that doesn't appear distance regular and the distance regular colorings of this graph, which are defined as the distance colorings with respect to completely regular codes. It is proved that the elements of the parameter matrix of an arbitrary distance regular coloring form two monotonic sequences. It is shown that every irreducible distance regular coloring of the n -dimensional rectangular grid has at most $2n + 1$ colors.

1 Introduction

A coloring of the vertices of a graph is perfect if any two vertices of the same color "see" the same number of vertices of any fixed color. If in addition the vertices are colored by distance from some initial set of vertices then the coloring is distance regular. This notion is closely related with distance regular graphs. In fact, in a distance regular graph the distance coloring with respect to an arbitrary vertex is perfect.

Complete classification of the perfect colorings of the 2-dimensional rectangular grid into 2, 3 and up to 9 colors can be found in [3], [10] and [7] respectively. All parameters of the distance regular colorings of the 2-dimensional rectangular grid were described in [1]. Parameters of perfect colorings with two colors of infinite circulant graphs were studied in [5, 6, 9]. The results of this paper were presented in part in [2].

For a distance regular graph the distance partition with respect to an arbitrary vertex is a perfect coloring; its parameters does not depend on the choice of the vertex. The completely regular codes in distance regular graphs are extensively investigated. We study the distance regular colorings in the graph of n -dimensional rectangular grid, which is not distance regular. We first prove the monotonicity of the upper and lower diagonals of the three-diagonal parameter matrix of a distance regular coloring (Theorem 1). Then we obtain that the number of colors does not exceed $2n + 1$ and show that this bound is attainable (Theorem 4).

Let's pass to the precise definitions. A k -coloring of the vertices of a graph can be presented as a function φ over the graph vertices with values in the set $\{1, 2, \dots, k\}$ and as a partition $\{C_1, C_2, \dots, C_k\}$ of the graph vertices, where $C_i = \{\mathbf{x} : \varphi(\mathbf{x}) = i\}$, $i = 1, 2, \dots, k$. We do not distinguish between these two interpretations. A k -coloring is *perfect* (in other terms, the partition is *equitable*) with the parameter

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matrix $A = (\alpha_{ij})_{k \times k}$ if any vertex of the color i has exactly α_{ij} adjacent vertices of color j for all $i, j \in \{1, 2, \dots, k\}$. A perfect coloring is *distance regular* if there exists an order (call, the *standard order*) of the colors such that the parameter matrix is three-diagonal with respect to this ordering. Note that exactly two color ordering of an arbitrary distance regular coloring are standard: the second one is inverse to the first. In what follows we suppose that the colors are numbered in the standard order. In other words, a perfect coloring is distance regular if the set of vertices of a color i , $i = 2, 3, \dots, k$, consists of all vertices at distance $i - 1$ from the set of vertices of the first color. This notion is intimately related to the notion of completely regular code. Actually, according to the definition of a completely regular code [8], the vertices of the first color (the last color) compose a completely regular code. When studying codes, we stress at the code distance and cardinality. When studying colorings, we regard at the code complement and stress at the parameter matrix and structure of all colors.

Denote nonzero elements of the parameter matrix of a distance regular coloring:

$l_i = \alpha_{i, i-1}$ ($i = 2, 3, \dots, k$) – the *lower degree* of the i -th color;

$k_i = \alpha_{i, i}$ ($i = 1, 2, \dots, k$) – the *inner degree* of the i -th color;

$u_i = \alpha_{i, i+1}$ ($i = 1, 2, \dots, k - 1$) – the *upper degree* of the i -th color.

In these terms, any vertex of color i "sees" l_i vertices of color $i - 1$, k_i vertices of color i and u_i vertices of color $i + 1$; obviously $l_i + k_i + u_i = 2n$ for any i . We will say that the color i has the *degree triple* (l_i, k_i, u_i) .

Let \mathbf{e}^i , $i = 1, \dots, n$, be the unit vector, i.e. $(0, 1)$ -vector with a unique one at the i -th position and $S(\mathbf{x})$ be the sphere of radius 1 centered at \mathbf{x} . Fix an arbitrary distance regular coloring φ of \mathbb{Z}^n and an arbitrary vertex $\mathbf{x} \in \mathbb{Z}^n$, $\varphi(\mathbf{x}) = i$. Let us introduce the following sets of unit vectors:

$$\begin{aligned} L_\varphi(\mathbf{x}) &= \{\mathbf{y} - \mathbf{x} : \varphi(\mathbf{y}) = i - 1, \mathbf{y} \in S(\mathbf{x})\}, \\ I_\varphi(\mathbf{x}) &= \{\mathbf{y} - \mathbf{x} : \varphi(\mathbf{y}) = i, \mathbf{y} \in S(\mathbf{x})\}, \\ U_\varphi(\mathbf{x}) &= \{\mathbf{y} - \mathbf{x} : \varphi(\mathbf{y}) = i + 1, \mathbf{y} \in S(\mathbf{x})\}. \end{aligned}$$

We omit the subscript φ if the coloring is clear from the context. We refer to the vectors in the sets $L_\varphi(\mathbf{x}), I_\varphi(\mathbf{x}), U_\varphi(\mathbf{x})$ as *lower*, *inner* and *upper directions* of the vertex \mathbf{x} with respect to φ . Obviously,

$$\begin{aligned} |L_\varphi(\mathbf{x})| &= l_i, \quad |I_\varphi(\mathbf{x})| = k_i, \quad |U_\varphi(\mathbf{x})| = u_i \quad \text{and} \\ L_\varphi(\mathbf{x}) \cup I_\varphi(\mathbf{x}) \cup U_\varphi(\mathbf{x}) &= \{\pm \mathbf{e}^i, \quad i = 1, \dots, n\}. \end{aligned}$$

We say that two colorings φ and ψ are *equivalent* if ψ can be obtained from φ by some translate and some color reordering. In particular, for a distance regular coloring φ , the coloring ψ with the inverse order of colors is equivalent and distance regular; moreover, for an arbitrary vertex \mathbf{x} ,

$$L_\varphi(\mathbf{x}) = U_\psi(\mathbf{x}), \quad U_\varphi(\mathbf{x}) = L_\psi(\mathbf{x}), \quad I_\varphi(\mathbf{x}) = I_\psi(\mathbf{x}). \quad (1)$$

For any set D of directions, use the notation $-D$ for the set $\{-d : d \in D\}$.

2 Reducible colorings

Consider colorings of 1-dimensional grid \mathbb{Z}^1 . For an arbitrary k , there exist only three nonequivalent perfect k -colorings; they are distance regular and periodical. We write

theirs periods as sequences of colors:

$$\begin{aligned} &1, 2, \dots, k-1, k, k-1, \dots, 2; \\ &1, 1, 2, \dots, k-1, k, k-1, \dots, 2; \\ &1, 1, 2, \dots, k-1, k, k, k-1, \dots, 2. \end{aligned}$$

A coloring $\varphi = \varphi(x_1, \dots, x_n)$ of \mathbb{Z}^n is called *reducible* if it can be reduced to the 1-dimensional coloring, i.e. there exists a k -coloring φ_1 of \mathbb{Z}^1 and $\delta_1, \dots, \delta_k \in \{0, 1, -1\}$ such that for any $(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$,

$$\varphi(x_1, x_2, \dots, x_n) = \varphi_1(\delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n). \quad (2)$$

Readily, if the coloring φ_1 of \mathbb{Z}^1 is distance regular then the coloring φ defined in accordance with (2) is also distance regular.

The parameter matrix of a perfect coloring is referred to as *reducible* if it admits a reducible coloring. All reducible matrices (obtained by (2)) have the form

$$\begin{pmatrix} 2n - \varepsilon_1 r & \varepsilon_1 r & \cdot & \cdot & \cdot & 0 & 0 \\ r & 2n - 2r & r & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & r & 2n - 2r & r \\ 0 & 0 & \cdot & \cdot & \cdot & \varepsilon_2 r & 2n - \varepsilon_2 r \end{pmatrix},$$

where r equals to the number of nonzero coefficients δ_i and $\varepsilon_1, \varepsilon_2 \in \{1, 2\}$ (the colorings with $(\varepsilon_1, \varepsilon_2) = (1, 2)$ and $(2, 1)$ are equivalent).

We call a coloring $\varphi : \mathbb{Z}^n \rightarrow \{1, \dots, k\}$ *cylindrical* if it is obtained from a coloring $\varphi' : \mathbb{Z}^m \rightarrow \{1, \dots, k\}$, $m < n$, by adding nonessential variables. Clearly, once the initial coloring φ' is perfect (distance regular) then the cylindrical coloring is also perfect (respectively, distance regular).

3 Upper and lower degrees

Fix an arbitrary distance regular k -coloring φ of \mathbb{Z}^n . We are going to prove the monotonicity of the upper degrees (and the lower degrees) of the coloring φ . Note that not every distance regular graph possesses this property.

Lemma 1. *For any i , $1 \leq i \leq k-1$, and any two adjacent vertices \mathbf{x} and \mathbf{y} of colors i and $i+1$ respectively we have $L(\mathbf{x}) \subseteq L(\mathbf{y})$ and $U(\mathbf{x}) \supseteq U(\mathbf{y})$.*

Proof. Let $d \in L(\mathbf{x})$; then the color of the vertex $\mathbf{z} = \mathbf{x} + d$ equals $i-1$. Since the coloring is distance regular, the vertex $\mathbf{v} \in \mathbf{y} + d$ has the color i , and then $d \in L(\mathbf{y})$. \square

We refer to the sequence of vertices $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^r \in \mathbb{Z}^n$ ($r \leq k$) as an *ascending chain*, if $\varphi(\mathbf{x}^i) = \varphi(\mathbf{x}^{i-1}) + 1$ and the distance between \mathbf{x}^{i-1} and \mathbf{x}^i equals 1 for $i = 2, \dots, r$. As a simple consequence of Lemma 1, we have

Corollary 1. *Let $r \leq k$, and let $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^r$ be an ascending chain. Then*

$$\begin{aligned} L(\mathbf{x}^1) &\subseteq L(\mathbf{x}^2) \subseteq \dots \subseteq L(\mathbf{x}^r), \\ U(\mathbf{x}^1) &\supseteq U(\mathbf{x}^2) \supseteq \dots \supseteq U(\mathbf{x}^r). \end{aligned}$$

So, we obtain the monotonicity of the lower degrees and the upper degrees:

Theorem 1. *For an arbitrary distance regular k -coloring of \mathbb{Z}^n , it holds*

$$l_2 \geq \dots \geq l_{k-1} \geq l_k \quad \text{and} \quad u_1 \leq u_2 \leq \dots \leq u_{k-1}.$$

It follows from Theorem 1 that the sequence of colors is partitioned into three segments $\{1, \dots, I_1\}$, $\{I_1 + 1, \dots, I_2 - 1\}$, $\{I_2, \dots, k\}$ (and the parameter matrix, into three layers):

in the first segment, for every $i \in \{1, \dots, I_1\}$, it holds $l_i < u_i$,
in the second one, for every $i \in \{I_1 + 1, \dots, I_2 - 1\}$, it holds $l_i = u_i$,
in the third, for every $i \in \{I_2, \dots, k\}$, it holds $l_i > u_i$.

We are going to prove that all lower degrees in the first segment are distinct, as well as the upper degrees in the last segment.

Lemma 2. *a) If $l_i = l_{i+1}$ then $u_i \leq l_i$ and $-U(\mathbf{x}) \subseteq L(\mathbf{x})$ for any vertex \mathbf{x} of color i .
b) If $u_i = u_{i-1}$ then $l_i \leq u_i$ and $-L(\mathbf{x}) \subseteq U(\mathbf{x})$ for any vertex \mathbf{x} of color i .*

Proof. Let $d \in U(\mathbf{x})$. Then the vertex $\mathbf{x} + d$ has the color $i + 1$. It follows from Lemma 1 that $L(\mathbf{x}) = L(\mathbf{x} + d)$. Hence, $-l \in L(\mathbf{x} + d) = L(\mathbf{x})$, and we get a). Now (1) gives b). \square

The next theorem is a simple consequence of Lemma 2:

Theorem 2. *For any $i < I_1$, it holds $l_i \neq l_{i+1}$; for any $i > I_2$, it holds $u_i \neq u_{i-1}$.*

In conclusion of this section, we state the following.

Corollary 2. *Let $i, j \in \{2, \dots, k-1\}$, $i \neq j$, and let the degree triples of colors i and j coincide. Then the degree triples of all colors between i and j have the form $(a, 2b, a)$.*

4 Colors with the same degree triples

Theorem 2 establishes that only the degree triple of form $(a, 2b, a)$ can be repeated. Everywhere until the end of this section we suppose $k \geq 4$ and $I_2 > I_1 + 2$; i.e., there exist repeated degree triples.

Lemma 3. *Let the colors i and $i + 1$ have the same degree triples. Then for any two adjacent vertices \mathbf{x} and \mathbf{y} of colors i and $i + 1$, respectively, we have*

$$L(\mathbf{x}) = L(\mathbf{y}) = -U(\mathbf{x}) = -U(\mathbf{y}),$$

$$I(\mathbf{x}) = I(\mathbf{y}) = -I(\mathbf{x}) = -I(\mathbf{y}).$$

Proof. First note that $L(\mathbf{x}) = L(\mathbf{y})$ and $U(\mathbf{x}) = U(\mathbf{y})$ by Lemma 1. Fix the direction $d \in L(\mathbf{x})$. Then $d \in L(\mathbf{y})$ by Lemma 1, this means that $-d \in U(\mathbf{y} + d)$. Using Lemma 1 with our condition $u_i = u_{i+1}$, we get $-d \in U(\mathbf{y})$. Again by Lemma 1, we obtain $-d \in U(\mathbf{x})$. The second equality follows. \square

We emphasize that according to the Lemma 3, two opposite directions belong or do not belong to the set of inner directions simultaneously.

Let us describe the set of vertices with the fixed color i . For any set of vertices $V \subseteq \mathbb{Z}^n$ denote by $G(V)$ the subgraph of \mathbb{Z}^n generated by V .

Lemma 4. *Let the colors i and $i+1$ have the same degree triples. Then for any two vertices \mathbf{x} and \mathbf{y} of a connected component of the graph $G(C_i \cup C_{i+1})$, we have*

$$L(\mathbf{x}) = L(\mathbf{y}) = -U(\mathbf{x}) = -U(\mathbf{y}),$$

$$I(\mathbf{x}) = I(\mathbf{y}) = -I(\mathbf{x}) = -I(\mathbf{y}).$$

Proof. It is sufficient to prove the equalities for two adjacent vertices of the same color i or $i+1$, for example for the color i , and then apply Lemma 3. Let us show that the second equation holds for every two adjacent vertices \mathbf{x} and \mathbf{y} of color i . Suppose that for some $d \in I(\mathbf{x})$ this direction is not inner for \mathbf{y} ; i.e., without loss of generality, $d \in U(\mathbf{x})$. Fig. 1 with the notations of vertices and Fig. 2 with theirs colors illustrate our reasoning. We reconstruct the colors in alphabetical order of vertices in Fig. 1 and get the coloring as in Fig. 2.

	$\mathbf{x} + d$	$\mathbf{y} + d$	e
a	\mathbf{x}	\mathbf{y}	b
	c	d	

Fig. 1

	i	$i+1$	$i+2$
i	i	i	i
	i	$i-1$	

Fig. 2

Colors i and $i+2$ of the adjacent vertices b and e come in to collision. The first equation follows from the second and Lemma 3. \square

Let us Clarify the structure of connected components of the graph $G(C_i \cup C_{i+1})$. Let $c \in \mathbb{Z}$ and $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \{0, 1, -1\}^n$. Denote by $M(\delta, c)$ the hyperplane

$$M(\delta, c) = \{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n : \delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n = c\}.$$

Lemma 5. *Let the colors i and $i+1$ have the same degree triples and let G' be a connected component of $G(C_i \cup C_{i+1})$. Then there exist integer c and $(0, 1, -1)$ -valued vector $\delta = (\delta_1, \dots, \delta_n)$ such that*

$$C_i \cap G' = M(\delta, c), \quad C_{i+1} \cap G' = M(\delta, c+1).$$

Proof. Let the repeated degree triple be $(t, 2n-2t, t)$. Let $\mathbf{v} = (v_1, v_2, \dots, v_n) \in G'$ be of color i . Without loss of generality, we suppose

$$U(\mathbf{v}) = \{\mathbf{e}^1, \dots, \mathbf{e}^s, -\mathbf{e}^{s+1}, \dots, -\mathbf{e}^t\},$$

$$L(\mathbf{v}) = \{-\mathbf{e}^1, \dots, -\mathbf{e}^s, \mathbf{e}^{s+1}, \dots, \mathbf{e}^t\},$$

$$I(\mathbf{v}) = \{\pm \mathbf{e}^{t+1}, \dots, \pm \mathbf{e}^n\}.$$

Then define the following constants:

$$\delta_1 = \dots = \delta_s = 1,$$

$$\delta_{s+1} = \dots = \delta_t = -1,$$

$$\delta_{t+1} = \dots = \delta_n = 0,$$

$$c = v_1 + \dots + v_s - v_{s+1} - \dots - v_t.$$

All neighbors of the vertices of G' belong to the set $G' \cup M(\delta, c-1) \cup M(\delta, c+2)$. For an arbitrary vertex $\mathbf{x} \in G'$, one can easily check by induction on distance between \mathbf{x} and \mathbf{v} that $\mathbf{x} \in M(\delta, c)$ in case $\mathbf{x} \in C_i$ and $\mathbf{x} \in M(\delta, c+1)$ in case $\mathbf{x} \in C_{i+1}$. \square

Theorem 3. *Let $\varphi : \mathbb{Z}^n \rightarrow \{1, 2, \dots, k\}$ be the distance regular coloring, and let for some $i, j \in \{2, \dots, k-2\}$, the colors i and j have the same degree triples. Then the degree triples coincide for all colors from 2 to $k-1$ and the coloring is reducible.*

Proof. By Lemma 5, there exists $c \in \mathbb{Z}$ and $\delta \in \{0, 1, -1\}^n$ such that $M(\delta, c + \varepsilon) \subseteq C_{i+\varepsilon}$, $\varepsilon \in \{0, 1\}$. Then for any $j \in \{1, \dots, k\}$ by induction on $|j - i|$ one can easily check that $M(\delta, c + j - i) \subseteq C_j$ because the coloring is distance regular. In particular, it holds $M(\delta, c - i + 1) \subseteq C_1$, $M(\delta, c + k - i) \subseteq C_k$. By distance regularity of the coloring $M(\delta, c - i) \subseteq C_1$ or $M(\delta, c + j - i) \subseteq C_2$ and $M(\delta, c + k - i) \subseteq C_k$ or $M(\delta, c + k - i) \subseteq C_{k-1}$. Finally, we find that

$$\varphi(x_1, x_2, \dots, x_n) = \varphi'(\delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n),$$

where φ' is a distance regular k -coloring of \mathbb{Z}^1 . □

5 The number of colors

Let us see how to derive distance regular colorings of the n -dimensional rectangular grid from the colorings of the $2n$ -dimensional Hamming space $\mathbf{F}^{2n} = \{0, 1\}^{2n}$. Let $g : \mathbf{F}^{2n} \rightarrow \mathbb{Z}^n$ be the Gray transform; i.e., for $\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbf{F}^{2n}$

$$g(\alpha_1, \dots, \alpha_{2n}) = (g_0(\alpha_1, \alpha_2), \dots, g_0(\alpha_{2n-1}, \alpha_{2n})),$$

where $g_0(00) = 0$, $g_0(01) = 1$, $g_0(11) = 2$, $g_0(10) = 3$. Let ψ be a coloring of \mathbf{F}^{2n} . Define the coloring φ of \mathbb{Z}^n as follows:

$$\varphi(x_1, \dots, x_n) = \psi(g^{-1}(x_1, \dots, x_n)), \quad (x_1, \dots, x_n) \in \mathbb{Z}^n. \quad (3)$$

Lemma 6. *Let ψ be a perfect (distance regular) coloring of \mathbf{F}^{2n} . Then the coloring φ of \mathbb{Z}^n defined by (3) is also perfect (respectively, distance regular) with the same parameter matrix.*

Proof. As the Gray transform preserves the adjacency, the statement is straightforward from the definitions. □

We take as ψ the distance coloring of \mathbf{F}^{2n} with respect to the all-zero vertex:

$$\psi(\mathbf{x}) = wt(\mathbf{x}) + 1, \quad \mathbf{x} \in \mathbf{F}^{2n}, \quad (4)$$

where $wt(\mathbf{x}) = \sum_{i=1}^{2n} x_i$ is the Hamming weight of the vertex \mathbf{x} . It is distance regular with the parameters $l_i = i - 1$, $u_i = 2n - i + 1$, $k_i = 0$, $i = 1, 2, \dots, 2n + 1$. Then, by Lemma 6, the coloring φ is also distance regular with the same parameters. The coloring φ is not reducible because its parameter matrix is not reducible. Moreover, all variables of φ are essential and the coloring is not cylindrical.

Finally, we can state the main theorem.

Theorem 4. *For an arbitrary irreducible distance regular k -coloring of n -dimensional rectangular grid, it holds $k \leq 2n + 1$. An irreducible distance regular $(2n + 1)$ -coloring exists.*

Proof. By Theorem 3, the coloring is irreducible and every two colors have different degree triples. This means that $I_2 - I_1 \leq 2$. Using Theorem 2, we get $I_1 \leq n$ and $k - I_2 + 1 \leq n$. Finally, $k \leq 2n + 1$. The coloring constructed above (??) gives us the example of the irreducible $(2n + 1)$ -coloring. □

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